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1987 J. Phys. A: Math. Gen. 20 6633

(http://iopscience.iop.org/0305-4470/20/18/062)

J. Phys. A: Math. Gen. **20** (1987) 6633-6634. Printed in the UK

ADDENDUM

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Received 27 July 1987

Abstract. Formulae for the amplitudes of each diagonal element in terms of the phases of the off-diagonal elements of the 3×3 unitary symmetric S matrix are given. These formulae are a great improvement and simplification on those given recently and have a form similar to the inverse relations. There are no problems with the choice of signs of square roots at any point within the Waldenstrøm pyramid.

In a recent paper (Kabir and Kermode 1987) we have shown that for the off-diagonal element of the three-channel S matrix

$$S_{ij} = N_{ij} \exp[\mathrm{i}(\delta_i + \delta_j + \beta_{ij})] \qquad i \neq j \tag{1}$$

the phase β_{ij} is simply related to the amplitude η_i of the diagonal element

$$S_{ii} = \eta_i \exp(2\mathrm{i}\delta_i) \qquad i = 1, 2, 3. \tag{2}$$

The relations are

$$\tan^2 \beta_{12} = AB/CD \qquad \tan^2 \beta_{13} = AD/BC \qquad \tan^2 \beta_{23} = AC/BD \tag{3}$$

where

$$\begin{aligned} A &= (\eta_1 + \eta_2 + \eta_3)^2 - 1 \qquad & B &= 1 - (\eta_1 + \eta_2 - \eta_3)^2 \\ C &= 1 - (\eta_2 + \eta_3 - \eta_1)^2 \qquad & D &= 1 - (\eta_1 + \eta_3 - \eta_2)^2. \end{aligned} \tag{4}$$

Waldenstrøm (1974) has given some inequalities for η_1, η_2 and η_3 and has shown that they lead to a region of (η_1, η_2, η_3)-space bounded by a pyramid. Points in or on the pyramid are allowed by the unitarity of the S matrix. From this pyramid, or from $0 \leqslant \eta_i \leqslant 1$, we can add some lower bounds to Waldenstrøm's inequalities. Thus

$$\begin{aligned} &\eta_1 + \eta_2 + \eta_3 \geqslant 1 \qquad & &-1 \leqslant \eta_1 + \eta_2 - \eta_3 \leqslant 1 \\ &-1 \leqslant \eta_2 + \eta_3 - \eta_1 \leqslant 1 \qquad & &-1 \leqslant \eta_1 + \eta_3 - \eta_2 \leqslant 1. \end{aligned} \tag{5}$$

It follows immediately that

$$A \geqslant 0 \qquad 0 \leqslant B \leqslant 1 \qquad 0 \leqslant C \leqslant 1 \qquad 0 \leqslant D \leqslant 1. \tag{6}$$

The formulae for the η_i in terms of the β_{ij} given in our previous paper do not hold for all points in the pyramid. For example, if the values $\eta_1 = \eta_2 = 0.1$, $\eta_3 = 1.0$ are used to calculate A, B, C and D and these formulae used to 'recalculate' η_1, we obtain $\eta_1 = 0.9$, rather than $\eta_1 = 0.1$. This follows because we assumed the lower bound in the inequalities (5) was 0 instead of -1. The formulae can be modified with a suitable self-consistency procedure to determine the appropriate sign for each square root. However, we shall not give these modifications because we have found simpler formulae, which cause no problems with square root ambiguities because B, C and D are non-negative (equation (6)) as are the η_i.

Writing

$$
\begin{aligned}
b &= B/A = |\cot \beta_{13} \cot \beta_{23}| \\
c &= C/A = |\cot \beta_{12} \cot \beta_{13}| \\
d &= D/A = |\cot \beta_{23} \cot \beta_{12}|
\end{aligned}
\tag{7}
$$

(note the introduction of the modulus sign) and letting

$$
\begin{aligned}
\lambda &= 1 - b + c + d = 8\eta_1\eta_2/A \geqslant 0 \\
\mu &= 1 + b - c + d = 8\eta_2\eta_3/A \geqslant 0 \\
\nu &= 1 + b + c - d = 8\eta_3\eta_1/A \geqslant 0
\end{aligned}
\tag{8}
$$

we have

$$
8\eta_1^2 = A\lambda\nu/\mu \qquad 8\eta_2^2 = A\lambda\mu/\nu \qquad 8\eta_3^2 = A\mu\nu/\lambda \tag{9}
$$

where $A = 1/t$ is obtained from

$$
t = (\lambda\mu + \mu\nu + \nu\lambda)^2/8\lambda\mu\nu - 1. \tag{10}
$$

Equations (9) and (10) are much simpler than the corresponding equations in the previous paper and there is no problem with the square roots because η_1, η_2 and η_3 are non-negative. Also, the form of equations (9) is similar to the form of equations (3), the inverse relations. We have checked these formulae by taking many points within the Waldenstrøm pyramid, calculating A, B, C and D and then 'recalculating' η_1, η_2 and η_3 from equations (9) and (10). The correct answers were obtained for each point considered.

AK is grateful to the FME, Nigeria, for financial support.

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